Neutrino electromagnetic properties: new approach to oscillations in magnetic fields

Alexander Dmitriev
Affiliation
Department of Theoretical Physics, Faculty of Physics, Lomonosov Moscow State
University, Moscow 119991, Russia
E-mail: ai.dmitriev@physics.msu.ru

Riccardo Fabbricatore
Affiliation
Department of Theoretical Physics, Faculty of Physics, Lomonosov Moscow State
University, Moscow 119991, Russia

E-mail: riccardo.fabbricatore1@gmail.com

Alexander Studenikin*
Affiliation
Department of Theoretical Physics, Faculty of Physics, Lomonosov Moscow State
University, Moscow 119991, Russia
Joint Institute for Nuclear Research, Dubna 141980, Moscow Region, Russia
E-mail: studenik@srd.sinp.msu.ru

Several new and interesting aspects of neutrino oscillations in a magnetic field are considered: 1) we develop a standard usually used approach to the neutrino spin oscillations in the neutrino mass basis and obtain the effective neutrino spin (and "spin-mass") oscillation Hamiltonian that can be used for description of the neutrino oscillations between different pairs of neutrino states with different masses and helicities; 2) we derive the exact solution of the Dirac equation for a massive neutrino with nonzero magnetic moment in the presence of a constant transversal magnetic field that is rotating along the direction of the neutrino propagation (the twisting magnetic field) and on the basis of the obtained energy spectrum the neutrino spin oscillation effective Hamiltonian is derive; 3) we develop a new approach to neutrino spin oscillations that is based on the description of the neutrino spin states with the corresponding spin operator that commutes with the neutrino dynamics Hamiltonian in the magnetic field. The obtained new results can have important phenomenological applications.

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*Speaker.

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1. Introduction

Experimental and theoretical studies of flavour conversion in solar, atmospheric, reactor and accelerator neutrino fluxes give strong evidence for nonzero neutrino masses. A massive neutrino can have nontrivial electromagnetic properties and, in particular, nonzero magnetic moment [1]. A detailed review on the neutrino electromagnetic interactions is given in [2]. If neutrinos have magnetic moments then in the presence of a magnetic field neutrinos spin can process [1, 3–7]. The possibility for the resonance amplification of the neutrino spin-flavour procession (if flavour neutrinos have the so-called transition magnetic moment) was first considered in [8, 9].

Neutrino transitions and oscillations in the presence of a magnetic field have been investigated before in different aspects. A detailed review of this topic is presented in [2] where the corresponding references to numerous publications can be also found. In our present paper we discuss several important issues in theoretical description of neutrino spin and spin-flavour oscillations. In Section 2 we develop a standard usually used approach to the neutrino spin oscillations in the neutrino mass basis and obtain the effective neutrino spin (and "spin-mass") oscillation Hamiltonian that can be used for description of the neutrino oscillations between different pairs of neutrino states with different masses and helicities. In Section 3 we derive the exact solution of the Dirac equation for a massive neutrino with nonzero magnetic moment in the presence of a constant transversal magnetic field that is rotating along the direction of the neutrino propagation (the twisting magnetic field). Using the obtained energy spectrum the neutrino spin oscillation effective Hamiltonian is derived. In Section 4 we develop a new approach to neutrino spin oscillations that is based on the description of the neutrino spin states with the corresponding spin operator that commutes with the neutrino dynamics Hamiltonian in the magnetic field. In the customary approach the neutrino helicity operator is used that does not commute with the neutrino dynamics Hamiltonian in the presence of a magnetic field.

2. Standard approach to neutrino spin oscillations in a magnetic field

Consider two physical neutrinos, ν_1 and ν_2, with corresponding masses m_1 and m_2. Each of the physical neutrinos are a superposition of the neutrino flavour states, ν_e and ν_μ. In general, a neutrino beam is a superposition of the mass states ν_1 and ν_2, or alternatively of the flavour states ν_e and ν_μ. Usually a notation for two neutrino basis are used,

$$\nu^{(p)} = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad \nu^{(f)} = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \tag{2.1}$$

and two neutrino basis are related by the mixing matrix

$$\nu^{(f)} = U\nu^{(p)}, \quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.2}$$

where θ is the mixing angle. We restrict ourselves to two neutrino generations. The evolution of the neutrino beam in the physical basis is governed by the Schrödinger type equation

$$i\frac{d}{dt}\nu^{(p)}(t) = H_{vac}\nu^{(p)}(t),\qquad(2.3)$$

where the Hamiltonian

$$H_{vac} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\qquad(2.4)$$

is diagonal in this basis, and the relativistic neutrino energies are

$$E_\alpha = \sqrt{\mathbf{p}^2 + m_\alpha^2} \approx |\mathbf{p}| + \frac{m_\alpha^2}{2|\mathbf{p}|}, \quad \alpha = 1,2.\qquad(2.5)$$

Consider neutrino oscillations in the presence of a constant magnetic field given by the vector $\boldsymbol{B}$. As it has been already mentioned in the introduction, a massive neutrino should have a nonzero magnetic moment. We introduce notations μ_{11} and μ_{22} for the corresponding magnetic moments of neutrinos ν_1 and ν_2. The magnetic moments interact with a magnetic field and from classical physics we know that the magnetic field exerts a torque on the magnetic moment. In quantum field theory, magnetic moment interaction between two neutrino fields ν_α and $\nu_{\alpha'}$ ($\alpha' = 1,2$) is given by

$$H_{EM} = -\frac{1}{2}\mu_{\alpha\alpha'}\overline{\nu}_{\alpha'}\sigma_{\mu\nu}\nu_\alpha F^{\mu\nu} + h.c. \; ,\qquad(2.6)$$

$F^{\mu\nu}$ is the electromagnetic field tensor. In a constant magnetic field the Hamiltonian (2.6) is

$$H_{EM} = -\frac{1}{2}\mu_{\alpha\alpha'}\overline{\nu}_{\alpha'}\boldsymbol{\Sigma}\boldsymbol{B}\nu_\alpha + h.c.,\qquad(2.7)$$

here

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},\qquad(2.8)$$

and σ_i are the Pauli matrices. When ν_α and $\nu_{\alpha'}$ represents different fields ($\alpha \neq \alpha'$) the values $\mu_{\alpha\alpha'}$ are called the transition magnetic moments.

Each of the neutrinos ν_1 and ν_2 can be in two spin states. These two neutrino quantum spin states with the same mass can be considered as two different neutrino species. Totally there are four neutrino species that can be mixed by the magnetic field presence and transitions between different neutrino spin states $\nu_{\alpha,s} \Leftrightarrow \nu_{\alpha',s'}$ ($\alpha, \alpha' = 1,2,\; s \neq s'$) can be generated. Here s and s' are two different neutrino spin quantum numbers. The probability of these conversions depend on time that provides the existence of neutrino spin oscillations. Note that there are two different types of neutrino spin oscillations that precede without changing of a neutrino mass state ($\alpha = \alpha'$) and with changing mass state ($\alpha \neq \alpha'$). The later case we call the neutrino spin-mass oscillations.

When a neutrino with nonzero magnetic moment is placed in an external magnetic field, different spin states acquire different magnetic potential energies which are similar to the matter potential [19] and must be accounted in the effective Hamiltonian in the Schrödinger type neutrino evolution equation (2.22) (see also [2]).

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Let us calculate the magnetic potential energies $V_{\alpha,s;\alpha',s'}$ corresponding to transitions between different neutrino states in the presence of an arbitrary oriented in respect to the neutrino propagation magnetic field. It is supposed (see Fig. 1) that neutrinos momenta are along $\mathbf{n_z}$ axis and $\mathbf{B}_\perp$ and $\mathbf{B}_\parallel$ are the transverse and longitudinal components of the magnetic field.
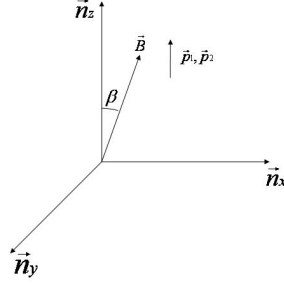

Рис. 1: An external magnetic field $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$ orientation in respect to the neutrino ν_1 and ν_2 momenta $\mathbf{p}_1$ and $\mathbf{p}_2$.

From the magnetic field interaction Hamiltonian (2.7) it follows that

$$V_{\alpha,s;\alpha',s'} = \langle \nu_{\alpha,s}|H_{EM}|\nu_{\alpha',s'}\rangle = -\frac{\mu_{\alpha,\alpha'}}{2}\int d^4x \nu_\alpha^\dagger \gamma_0 \begin{pmatrix} \boldsymbol{\Sigma B} & 0 \\ 0 & \boldsymbol{\Sigma B} \end{pmatrix} \nu_{\alpha'}, \tag{2.9}$$

where

$$\nu_{\alpha,s} = C_\alpha \sqrt{\frac{E_\alpha + m_\alpha}{2E_\alpha}} \begin{pmatrix} u_s \\ \frac{\boldsymbol{\Sigma p}_\alpha}{E_\alpha + m_\alpha} u_s \end{pmatrix} e^{-iE_\alpha t + i\boldsymbol{p}_\alpha \boldsymbol{x}}, \tag{2.10}$$

and

$$\nu_{\alpha',s'}(m_{\alpha'}, p_{\alpha'}, E_{\alpha'}) = C_{\alpha'} \sqrt{\frac{E_{\alpha'} + m_{\alpha'}}{2E_{\alpha'}}} \begin{pmatrix} u_{s'} \\ \frac{\boldsymbol{\Sigma p}_{\alpha'}}{E_{\alpha'} + m_{\alpha'}} u_{s'} \end{pmatrix} e^{-iE_{\alpha'} t + i\boldsymbol{p}_{\alpha'} \boldsymbol{x}}. \tag{2.11}$$

For the considered case of ultrarelativistic neutrinos we describe neutrino spin states by their helicities. Therefore for a particular type of a massive neutrino ν_α there are two different spin states given by

$$u_{s=1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_{s=-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.12}$$

that correspond to the right-handed ν_R and left-handed ν_L chiral neutrinos, respectively. As well known [1,3,5,6], the spin of such a neutrino can precess in a transversal magnetic field.

From (2.9) we get

$$V_{\alpha,s;\alpha',s'} = -\frac{1}{2}\mu_{\alpha\alpha'}C_\alpha C_{\alpha'} \int d^4x \boldsymbol{B} \left(u_s^\dagger \;\; \frac{\boldsymbol{\Sigma p}_\alpha}{E_\alpha + m_\alpha} u_s^\dagger \right) \begin{pmatrix} \boldsymbol{\Sigma} & 0 \\ 0 & -\boldsymbol{\Sigma} \end{pmatrix} \begin{pmatrix} u_{s'} \\ \frac{\boldsymbol{\Sigma p}_{\alpha'}}{E_{\alpha'} + m_{\alpha'}} u_{s'} \end{pmatrix}$$

$$\times \frac{\sqrt{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})}}{2\sqrt{E_\alpha E_{\alpha'}}} exp\left(-i\Delta Et + i\Delta px\right). \quad (2.13)$$

It is possible to show that

$$\boldsymbol{B} \left(u_s^\dagger \;\; \frac{\boldsymbol{\Sigma p}_\alpha}{E_\alpha + m_\alpha} u_s^\dagger \right) \begin{pmatrix} \boldsymbol{\Sigma} & 0 \\ 0 & -\boldsymbol{\Sigma} \end{pmatrix} \begin{pmatrix} u_{s'} \\ \frac{\boldsymbol{\Sigma p}_{\alpha'}}{E_{\alpha'} + m_{\alpha'}} u_{s'} \end{pmatrix} =$$

$$= u_s^\dagger \left((\boldsymbol{\Sigma B}_\parallel) \left(1 - \frac{\boldsymbol{p}_\alpha \boldsymbol{p}_{\alpha'}}{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})} \right) + \boldsymbol{\Sigma B}_\perp \left(1 + \frac{\boldsymbol{p}_\alpha \boldsymbol{p}_{\alpha'}}{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})} \right) \right) u_{s'}. \quad (2.14)$$

In the case of ultrarelativistic neutrinos $\frac{m_\alpha}{E_\alpha} \ll 1$ we have

$$\left(1 - \frac{\boldsymbol{p}_\alpha \boldsymbol{p}_{\alpha'}}{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})} \right) \frac{\sqrt{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})}}{2\sqrt{E_\alpha E_{\alpha'}}} = \gamma_{\alpha\alpha'}^{-1}, \quad (2.15)$$

where

$$\gamma_{\alpha\alpha'}^{-1} = \frac{1}{2} \left(\frac{m_\alpha}{E_\alpha} + \frac{m_{\alpha'}}{E_{\alpha'}} \right). \quad (2.16)$$

and

$$\left(1 + \frac{\boldsymbol{p}_\alpha \boldsymbol{p}_{\alpha'}}{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})} \right) \frac{\sqrt{(E_\alpha + m_\alpha)(E_{\alpha'} + m_{\alpha'})}}{2\sqrt{E_\alpha E_{\alpha'}}} = 1. \quad (2.17)$$

In the strightforward calculations we obtain that

$$u_{s=1}^\dagger \boldsymbol{\Sigma B} u_{s=1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{\Sigma} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\boldsymbol{B}_\parallel + \boldsymbol{B}_\perp) =$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \sigma_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} B\cos\beta + \begin{pmatrix} 1 & 0 \end{pmatrix} \sigma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} B\sin\beta = B\cos\beta, \quad (2.18)$$

$$u_{s=1}^\dagger \boldsymbol{\Sigma B} u_{s=-1} = B\sin\beta, \quad (2.19)$$

$$u_{s=-1}^\dagger \boldsymbol{\Sigma B} u_{s=1} = B\sin\beta, \quad (2.20)$$

$$u_{s=-1}^\dagger \boldsymbol{\Sigma B} u_{s=-1} = -B\cos\beta. \quad (2.21)$$

As it can be expected, in neutrino transitions without change of helicity only the $B_\parallel = B\cos\beta$ component of the magnetic field contribute to the effective potential, whereas in transitions with change of the neutrino helicity the transversal component $B_\perp = B\sin\beta$ matters.

Now we consider two different neutrino mass states with two helicities as four neutrino species $\nu_{1,s=1}$, $\nu_{1,s=-1}$, $\nu_{2,s=1}$ and $\nu_{2,s=-1}$. From the performed calculations we conclude that

the effective neutrino oscillation Hamiltonian that governs the massive neutrino oscillations in the presence of a constant magnetic field is

$$H_{em} = H_{vac} + H_\mu, \tag{2.22}$$

where the corresponding contributions of the magnetic potentials $V_{\alpha,s;\alpha',s'}$ to the oscillation Hamiltonian are given by

$$H_\mu = \frac{1}{2} \begin{pmatrix} \mu_{11}\frac{B_\parallel}{\gamma_{11}} & \mu_{11}B_\perp & \mu_{12}\frac{B_\parallel}{\gamma_{12}} & \mu_{12}B_\perp \\ \mu_{11}B_\perp & -\mu_{11}\frac{B_\parallel}{\gamma_{11}} & \mu_{12}B_\perp & -\mu_{12}\frac{B_\parallel}{\gamma_{12}} \\ \mu_{12}\frac{B_\parallel}{\gamma_{12}} & \mu_{12}B_\perp & \mu_{22}\frac{B_\parallel}{\gamma_{22}} & \mu_{22}B_\perp \\ \mu_{12}B_\perp & -\mu_{12}\frac{B_\parallel}{\gamma_{12}} & \mu_{22}B_\perp & -\mu_{22}\frac{B_\parallel}{\gamma_{22}} \end{pmatrix}. \tag{2.23}$$

Finally, for the neutrino evolution equation we get

$$i\frac{d}{dt}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{1,s=-1} \\ \nu_{2,s=1} \\ \nu_{2,s=-1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} E_1 + \mu_{11}\frac{B_\parallel}{\gamma_{11}} & \mu_{11}B_\perp & \mu_{12}\frac{B_\parallel}{\gamma_{12}} & \mu_{12}B_\perp \\ \mu_{11}B_\perp & E_1 - \mu_{11}\frac{B_\parallel}{\gamma_{11}} & \mu_{12}B_\perp & -\mu_{12}\frac{B_\parallel}{\gamma_{12}} \\ \mu_{12}\frac{B_\parallel}{\gamma_{12}} & \mu_{12}B_\perp & E_2 + \mu_{22}\frac{B_\parallel}{\gamma_{22}} & \mu_{22}B_\perp \\ \mu_{12}B_\perp & -\mu_{12}\frac{B_\parallel}{\gamma_{12}} & \mu_{22'}B_\perp & E_2 - \mu_{22}\frac{B_\parallel}{\gamma_{22}} \end{pmatrix}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{1,s=-1} \\ \nu_{2,s=1} \\ \nu_{2,s=-1} \end{pmatrix}. \tag{2.24}$$

The obtained neutrino evolution equation contains information on all possible oscillations in the presence of a magnetic field between different pairs from four neutrino states with masses m_1 and m_2 and helicities given by $s = 1$ and $s = -1$. Consider three typical cases. For neutrino evolution between two neutrino states with equal masses and different helicities, for instance $\nu_{1,s=1} \Leftrightarrow \nu_{1,s=-1}$, from (2.24) it follows

$$i\frac{d}{dt}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{1,s=-1} \end{pmatrix} = \frac{\mu_{11}}{2}\begin{pmatrix} \frac{B_\parallel}{\gamma_{11}} & B_\perp \\ B_\perp & -\frac{B_\parallel}{\gamma_{11}} \end{pmatrix}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{1,s=-1} \end{pmatrix}. \tag{2.25}$$

For neutrino evolution between two neutrino states with different masses and different helicities, for instance $\nu_{1,s=1} \Leftrightarrow \nu_{2,s=-1}$, from (2.24) it follows

$$i\frac{d}{dt}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{2,s=-1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\Delta m^2}{4E} + \mu_{11}\frac{B_\parallel}{\gamma_{11}} & \mu_{12}B_\perp \\ \mu_{12}B_\perp & -\frac{\Delta m^2}{4E} - \mu_{22}\frac{B_\parallel}{\gamma_{22}} \end{pmatrix}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{2,s=-1} \end{pmatrix}. \tag{2.26}$$

For neutrino evolution between two neutrino states with different masses and equal helicities, for instance $\nu_{1,s=1} \Leftrightarrow \nu_{2,s=1}$, from (2.24) it follows

$$i\frac{d}{dt}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{2,s=1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\Delta m^2}{4E} + \mu_{11}\frac{B_\parallel}{\gamma_{11}} & \mu_{12}\frac{B_\parallel}{\gamma_{12}} \\ \mu_{12}\frac{B_\parallel}{\gamma_{12}} & -\frac{\Delta m^2}{4E} + \mu_{22}\frac{B_\parallel}{\gamma_{22}} \end{pmatrix}\begin{pmatrix} \nu_{1,s=1} \\ \nu_{2,s=1} \end{pmatrix}. \tag{2.27}$$

From the obtained equations (2.25), (2.26) and (2.27) several general conclusions on the influence of an arbitrary magnetic field $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$ on neutrino evolution are strightforward:

1) the mixing between two neutrinos with different helicities is due to the corresponding magnetic moment (or transition magnetic moment) interaction with the transversal magnetic field $B_\perp$,

2) the longitudinal magnetic field component $B_\parallel$ coupled to the corresponding magnetic moment shifts the neutrino energy,

3) in case of nonzero transition magnetic moment μ_{12}, the mixing between neutrino states with different masses is induced by the interaction with $B_\parallel$.

The obtained results can be translated to the flavour neutrino evolution and oscillations by inclusion of the neutrino mixing effects following to (2.2). Then the effect of the background matter can be also added. Note that a possibility of the resonant amplification of neutrino oscillations in the longitudinal magnetic field were discussed in [10–12].

3. Neutrino propagation and spin oscillations in twisting magnetic field

The problem of neutrino spin oscillations in the presence of a transversal in respect to the neutrino propagation and twisting magnetic field, given by

$$\mathbf{B}_\perp(z) = B_\perp \cos(kz)\mathbf{e}_x + B_\perp \sin(kz)\mathbf{e}_y, \tag{3.1}$$

where $k = 2\pi/\lambda$ and λ is the length of a period of the magnetic field rotation, and its possible phenomenological implications was discussed before (see, for instance, [12–17]). In this section we derive the exact solution for the Dirac equation for a massive neutrino with nonzero magnetic moment including the exact expression for the neutrino energy. Then on this basis we consider the neutrino spin and spin-mass oscillation in such a magnetic field.

We consider the Dirac equation for a massive neutrino moving in the twisting magnetic field in the following form,

$$\left[\gamma p - m_i - \mu_i B_\perp \left(\Sigma_1 \cos(kz) + \Sigma_2 \cos(kz)\right)\right] v_i(p) = 0, \tag{3.2}$$

where matrices are are given by (2.8). It is supposed that the neutrino propagates along $\mathbf{e}_z$ direction. Then the solution can be found in the form

$$\psi(z) = Ce^{-izp_3} \begin{pmatrix} C_1 e^{ikz/2} \\ C_2 e^{-ikz/2} \\ C_3 e^{ikz/2} \\ C_4 e^{-ikz/2} \end{pmatrix}. \tag{3.3}$$

From (3.2) and (3.3) for the neutrino wave function we obtain the following equation

$$\begin{pmatrix} (m_i - E) & \mu_i B_\perp e^{-ikz} & \hat{p}_3 & 0 \\ \mu_i B_\perp e^{ikz} & (m_i - E) & 0 & -\hat{p}_3 \\ \hat{p}_3 & 0 & -(m_i + E) & -\mu_i B_\perp e^{-ikz} \\ 0 & -\hat{p}_3 & -\mu_i B_\perp e^{ikz} & -(m_i + E) \end{pmatrix} \psi = 0, \tag{3.4}$$

and then

$$\begin{pmatrix} (m_i - E) & \mu_i B_\perp e^{-ikz} & (p_3 + k/2) & 0 \\ \mu_i B_\perp e^{ikz} & (m_i - E) & 0 & -(p_3 - k/2) \\ (p_3 + k/2) & 0 & -(m_i + E) & -\mu_i B_\perp e^{-ikz} \\ 0 & -(p_3 - k/2) & -\mu_i B_\perp e^{ikz} & -(m_i + E) \end{pmatrix} \psi = 0. \tag{3.5}$$

For the neutrino energy spectrum from (3.5) we obtain

$$E = \pm\sqrt{m_i^2 + p^2 + (k/2)^2 + (\mu_i B_\perp)^2 + 2s\sqrt{m_i^2(\mu_i B_\perp)^2 + p^2((\mu_i B_\perp)^2 + (k/2)^2)}}, \qquad (3.6)$$

where $s = \pm 1$ correspond to two neutrino helicity states. In the limits $p >> m_i$, $p >> \mu_i B_\perp$, $p >> k$ we can simplify it to

$$E = \pm\sqrt{m_i^2 + (p + s\sqrt{(\mu_i B_\perp)^2 + (k/2)^2})^2}. \qquad (3.7)$$

Finally, for the neutrino wave function components we obtain

$$\psi = C \begin{pmatrix} 1 \\ \chi\mu_i B_\perp \\ \frac{(E-m_i)+(\mu_i B_\perp)^2\chi}{p+\frac{k}{2}} \\ \frac{1+(E-m_i)\chi}{p-\frac{k}{2}}\mu_i B_\perp \end{pmatrix} e^{-i(Et-(p+\frac{k}{2})z)}, \qquad (3.8)$$

where

$$\chi = \frac{2(pE + m_i(k/2))/(p^2 - (k/2)^2)}{(p - k/2) + \frac{m_i^2 - E^2}{p - k/2} - \frac{(\mu_i B_\perp)^2}{p + k/2}}. \qquad (3.9)$$

In the case of relativistic neutrinos from (3.7) for the energy we get

$$E \approx p + \frac{m_i^2 + (k/2)^2 + (\mu_i B_\perp)^2}{2p} \pm \sqrt{(\mu_i B_\perp)^2 + (k/2)^2}. \qquad (3.10)$$

Consider two neutrino species $\nu_{s=+1}$ and $\nu_{s=-1}$ with equal masses $m = m_i$ and of different spin states $s = \pm 1$ that forms a new neutrino basis

$$\nu = \begin{pmatrix} \nu_{s=+1} \\ \nu_{s=-1} \end{pmatrix}. \qquad (3.11)$$

Obviously, the effective neutrino evolution Hamiltonian in this basis is diagonal and the evolution equation is

$$i\frac{d}{dt}\begin{pmatrix} \nu_{s=+1} \\ \nu_{s=-1} \end{pmatrix} = \begin{pmatrix} +\sqrt{(\mu B_\perp)^2 + (k/2)^2} & 0 \\ 0 & -\sqrt{(\mu B_\perp)^2 + (k/2)^2} \end{pmatrix}\begin{pmatrix} \nu_{s=+1} \\ \nu_{s=-1} \end{pmatrix}, \qquad (3.12)$$

here μ is the neutrino magnetic moment. In the case of two neutrino species $\nu_{1,s=+1}$ and $\nu_{2,s=-1}$ with different masses $m_1 \neq m_2$ and different magnetic moments μ_1 and μ_2 the evolution equation takes the form

$$i\frac{d}{dt}\begin{pmatrix} \nu_{1,s=+1} \\ \nu_{2,s=-1} \end{pmatrix} = \begin{pmatrix} \frac{\Delta m^2}{4p} + a_1 & 0 \\ 0 & -\frac{\Delta m^2}{4p} - a_2 \end{pmatrix}\begin{pmatrix} \nu_{1,s=+1} \\ \nu_{2,s=-1} \end{pmatrix}, \qquad (3.13)$$

where

$$a_i = \sqrt{(\mu_i B_\perp)^2 + (k/2)^2}. \qquad (3.14)$$

In the production and detection processes neutrino participate by their left-handed chiral (negative helicity) states. Therefore, in order to study possible phenomenological consequences of neutrino oscillations in twisting magnetic fields one should represent the initial left-handed neutrino as a linear combination of the basis neutrino states ν_{is}, $\nu_{js'}$ $(i,j = 1,2$ and $s,s' = \pm 1)$, then include the neutrino oscillation process that proceeds following the neutrino evolution equation (3.12) or (3.13), and at the final stage again extract the left-handed neutrino component that only interacts in the detector. The above scheme is implemented and discussed in details in the next section.

4. Neutrino oscillations within precise description of spin states in magnetic field

In this section we show that there can be more precise approach to neutrino oscillations in the presence of a magnetic field than one usually used in literature and that we also apply in Section 2. Within this customary approach the helicity operator is used for classification of a neutrino spin states in a magnetic field. However, the helicity operator does not commute with the neutrino Hamiltonian. This case resembles situation of the flavour neutrino oscillations in the nonadiabatic case when the neutrino mass states are not stationary. The proposed alternative approach to neutrino spin oscillations is based on the exact solutions of the corresponding Dirac equation for a massive neutrino wave function in the presence of a magnetic field that stipulates the description of the neutrino spin states with the corresponding spin operator that commutes with the neutrino dynamic Hamiltonian in the magnetic field.

Here we again consider a simple model with two generations of flavour neutrinos ν_e and ν_μ that are the orthogonal superpositions of mass states ν_1 and ν_2

$$\nu_f = \sum_i U_{fi}\nu_i, \tag{4.1}$$

where U_{fi} are elements the mixing matrix given by (2.2) and $f = e, \mu$, $i = 1,2$. We start with consideration of a massive ν_i with the magnetic moment μ_i that propagates along $\mathbf{n_z}$ direction in presence of constant homogeneous arbitrary oriented magnetic field $\mathbf{B} = (B_\perp, 0, B_\parallel)$. The neutrino wave function in the momentum representation is given by a plane wave solution of the modified Dirac equation

$$(\gamma p - m_i - \mu_i \mathbf{\Sigma B})\nu_i(p) = 0. \tag{4.2}$$

The neutrino energy spectrum can be determined from the condition

$$\det(\gamma p - m_i - \mu_i \mathbf{\Sigma B}) = 0, \tag{4.3}$$

which guarantees the existence of a nontrivial solution of the modified Dirac equation (4.2). For the neutrino energy spectrum we obtain

$$E_i^\pm = \sqrt{m_i^2 + p^2 + \mu_i{}^2 \mathbf{B}^2 \pm 2\mu_i\sqrt{m_i^2 \mathbf{B}^2 + p^2 B_\perp^2}}, \tag{4.4}$$

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where "$\pm$" denotes two different eigenvalues of the Hamiltonian

$$H_i = \gamma_0(m_i + \boldsymbol{\gamma}\mathbf{p} + \mu_i\boldsymbol{\Sigma}\mathbf{B}), \tag{4.5}$$

which describes dynamics of the neutrino system under consideration.

Of course such a simple model can pretend to an approximate description of a realistic physical process. It also enables one to reveal general properties of spin and spin-flavor oscillations in the presence of an arbitrary magnetic field that should be important for phenomenological applications.

Our approach is based, at first, on the obtained the exact neutrino energy spectrum (4.4) in the magnetic field. Secondly, we define different neutrino spin states in the mass basis as eigenstates of the spin operator (see [18])

$$S_i = \frac{1}{|\mathbf{B}|}\left(\boldsymbol{\Sigma}\mathbf{B} - \frac{i}{m_i}\gamma_0\gamma_5\left[\boldsymbol{\Sigma}\times\mathbf{p}\right]\mathbf{B}\right), \tag{4.6}$$

which commutes with the Hamiltonian (4.5). Hence, we specify the neutrino spin states as the stationary states for the Hamiltonian, contrary to the case when the helicity operator is used.

Consider the mass state ν_i as a superposition of neutrinos ν_i^+ and ν_i^- in a definite spin state,

$$\nu_i = c_i^+\nu_i^+ + c_i^-\nu_i^-. \tag{4.7}$$

The complex coefficients denote two different eigenstates of the spin operator S_i and $|c_i^+|^2 + |c_i^-|^2 = 1$. Thus, the neutrino mass states evolve following to

$$\nu_i(x) = \left[c_i^+ e^{-iE_i^+ t}\xi_i^+ + c_i^- e^{-iE_i^- t}\xi_i^-\right]e^{i\mathbf{p}\mathbf{x}}, \tag{4.8}$$

where the neutrino initial state at $t = 0$ is given by

$$\nu_i^\pm(t=0) = \xi_i^\pm e^{i\mathbf{p}\mathbf{x}}. \tag{4.9}$$

In the following calculations the term $e^{i\mathbf{p}\mathbf{x}}$ is neglected because it is irrelevant for the neutrino oscillation probability.

Next we assume that the initial neutrino state $\nu(t=0)$ is a pure electron state which is defined as the superposition of the mass states,

$$\nu(t=0) = \left[c_1^+\xi_1^+ + c_1^-\xi_1^-\right]\cos\theta + \left[c_2^+\xi_2^+ + c_2^-\xi_2^-\right]\sin\theta. \tag{4.10}$$

Using 4.8 we see that this state depends on time as

$$\nu(t) = \left[c_1^+ e^{-iE_1^+ t}\xi_1^+ + c_1^- e^{-iE_1^- t}\xi_1^-\right]\cos\theta + \left[c_2^+ e^{-iE_2^+ t}\xi_2^+ + c_2^- e^{-iE_2^- t}\xi_2^-\right]\sin\theta. \tag{4.11}$$

Therefore, the probability to observe the muon neutrino state ν_μ at time t is given by

$$P_{\nu_e \to \nu_\mu}(t) = \left|\left\langle\nu_\mu|\nu(t)\right\rangle\right|^2, \tag{4.12}$$

where

$$\nu_\mu = -\left[c_1^+\xi_1^+ + c_1^-\xi_1^-\right]\sin\theta + \left[c_2^+\xi_2^+ + c_2^-\xi_2^-\right]\cos\theta.$$

It is clear that

$$\xi_i^{s\dagger}\xi_i^{s'} = \delta^{ss'} \text{and} \quad \xi_2^{s\dagger}\xi_1^{s'} = \xi_1^{s\dagger}\xi_2^{s'} = 0, \tag{4.13}$$

where $s, s' = \pm$. Using (4.10) and (4.11) we get

$$\left|\langle\nu_\mu|\nu(t)\rangle\right| = \left(|c_2^+|^2 e^{-iE_2^+ t} + |c_2^-|^2 e^{-iE_2^- t} - |c_1^+|^2 e^{-iE_1^+ t} - |c_1^-|^2 e^{-iE_1^- t}\right)\sin\theta\cos\theta, \tag{4.14}$$

and

$$P_{\nu_e \to \nu_\mu}(t) = \left\{-|c_2^+|^2|c_2^-|^2 \sin^2\frac{E_2^+ - E_2^-}{2}t + |c_2^+|^2|c_1^+|^2 \sin^2\frac{E_2^+ - E_1^+}{2}t + \right.$$

$$+ |c_2^+|^2|c_1^-|^2 \sin^2\frac{E_2^+ - E_1^-}{2}t + |c_2^-|^2|c_1^+|^2 \sin^2\frac{E_2^- - E_1^+}{2}t +$$

$$\left. + |c_2^-|^2|c_1^-|^2 \sin^2\frac{E_2^- - E_1^-}{2}t - |c_1^+|^2|c_1^-|^2 \sin^2\frac{E_1^+ - E_1^-}{2}t\right\}\sin^2 2\theta. \tag{4.15}$$

It is usually assumed that the initial state of relativistic neutrino is a negative-helicity state, which means that

$$\frac{\mathbf{\Sigma p}}{|\mathbf{p}|}\xi_i = -\xi_i. \tag{4.16}$$

Next we consider the left-handed spinors because only the left-handed fermions participate in the production and detection processes and we suppose that each of the mass states of the initial electron neutrino are left-handed. In our case the helicity operator is equal to $\mathbf{\Sigma p}/|\mathbf{p}| = \sigma_3$, therefore the initial neutrino state is given by

$$\psi_L = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.17}$$

Let us write the initial neutrino state ψ_L as a superposition of the eigenvectors of the spin operator S_i. From (4.6) we get

$$S_i = \begin{pmatrix} \cos\phi & \sin\phi & 0 & -\frac{p}{m_i}\sin\phi \\ \sin\phi & -\cos\phi & \frac{p}{m_i}\sin\phi & 0 \\ 0 & \frac{p}{m_i}\sin\phi & \cos\phi & \sin\phi \\ -\frac{p}{m_i}\sin\phi & 0 & \sin\phi & -\cos\phi \end{pmatrix}, \tag{4.18}$$

where ϕ is the angle between $\mathbf{B}$ and $\mathbf{p}$. It is obvious that

$$S_i^2 = \left(1 + \frac{p^2}{m_i^2}\sin^2\phi\right)\hat{I}_{4\times 4}. \tag{4.19}$$

In order to define the spin projector operators we introduce the normalized spin operator following to

$$\tilde{S}_i = \sqrt{\frac{1}{1 + \frac{p^2}{m_i^2}\sin^2\phi}}S_i \equiv N_i S_i, \quad \tilde{S}_i^2 = 1. \tag{4.20}$$

The spin projector operators are

$$P_i^\pm = \frac{1 \pm \tilde{S}_i}{2}, \tag{4.21}$$

and we use them to split the initial neutrino state ψ_L in two neutrino states with definite spin quantum numbers

$$\psi_i^+ = P_i^+ \psi_L = \frac{1}{2} \begin{pmatrix} N_i \sin\phi \\ 1 - N_i \cos\phi \\ \frac{p}{m_i} N_i \sin\phi \\ 0 \end{pmatrix}, \quad \psi_i^- = P_i^- \psi_L = \frac{1}{2} \begin{pmatrix} -N_i \sin\phi \\ 1 + N_i \cos\phi \\ -\frac{p}{m_i} N_i \sin\phi \\ 0 \end{pmatrix}. \tag{4.22}$$

Note that $\psi_i = \psi_i^+ + \psi_i^- = c_i^+ \eta_i^+ + c_i^- \eta_i^-$, where $\eta_i^\pm$ is a basis in the spin operator S_i eigenspace. From the condition

$$\psi_i^{\dagger\pm} \psi_L^\pm = |c_i^\pm|^2 \tag{4.23}$$

we get that

$$|c_i^+|^2 = \frac{1 - N_i \cos\phi}{2}, \qquad |c_i^-|^2 = \frac{1 + N_i \cos\phi}{2}, \tag{4.24}$$

where as in eq.(4.20)

$$N_i = \sqrt{\frac{1}{1 + \frac{p^2}{m_i^2} \sin^2\phi}}. \tag{4.25}$$

Now we can insert the obtained expressions (4.24) for $|c_i^\pm|^2$ in eq. (4.15). In the forthcoming evaluation of the probability $P_{\nu_e \to \nu_\mu}(t)$ we consider the case when the magnetic field $\mathbf{B}$ is nearly a transversal one and $B_\perp \gg B_\parallel$, therefore

$$\sin\phi \approx 1, \quad \cos\phi \approx 0. \tag{4.26}$$

Then we get (it is also supposed that $\frac{p^2}{m_i^2} \gg 1$)

$$N_i \approx 1 - \frac{\frac{p^2}{2m_i^2} \sin^2\phi}{1 + \frac{p^2}{m_i^2} \sin^2\phi} \approx \frac{m_i^2}{p \sin^2\phi}. \tag{4.27}$$

Thus, for typical combinations of the coefficients $|c_i^\pm|^2$ of eq.(4.15) in the linear approximation over $\cos\phi \ll 1$ we get

$$|c_i^+|^2 |c_i^-|^2 = \frac{1 - N_i^2 \cos^2\phi}{4} \approx \frac{1}{4}, \tag{4.28}$$

$$|c_2^s|^2 |c_1^s|^2 \approx \frac{1}{4} \left(1 - s(N_1 + N_2)\cos\phi\right) \approx \frac{1}{4} \left(1 - s \frac{m_1^2 + m_2^2}{p^2} \cos\phi\right), \tag{4.29}$$

$$|c_2^+|^2 |c_1^-|^2 \approx \frac{1}{4} \left(1 - (N_2 - N_1)\cos\phi\right) \approx \frac{1}{4} \left(1 - \frac{m_2^2 - m_1^2}{p^2} \cos\phi\right), \tag{4.30}$$

$$|c_2^-|^2 |c_1^+|^2 \approx \frac{1}{4} \left(1 - (N_1 - N_2)\cos\phi\right) \approx \frac{1}{4} \left(1 - \frac{m_1^2 - m_2^2}{p^2} \cos\phi\right). \tag{4.31}$$

Using these expressions finally from (4.15) we get

$$P_{\nu_{eL}\to\nu_\mu}(t) = \frac{1}{4}\sin^2 2\theta \left\{ -\sin^2\frac{E_2^+ - E_2^-}{2}t + \sin^2\frac{E_2^+ - E_1^+}{2}t + \sin^2\frac{E_2^+ - E_1^-}{2}t + \right.$$
$$\left. + \sin^2\frac{E_2^- - E_1^+}{2}t + \sin^2\frac{E_2^- - E_1^-}{2}t - \sin^2\frac{E_1^+ - E_1^-}{2}t \right\} +$$
$$+ \frac{m_1^2 + m_2^2}{p^2}\sin^2 2\theta\cos\phi\left\{\sin^2\frac{E_2^- - E_1^-}{2}t - \sin^2\frac{E_2^+ - E_1^+}{2}t\right\} +$$
$$+ \frac{m_2^2 - m_1^2}{p^2}\sin^2 2\theta\cos\phi\left\{\sin^2\frac{E_2^- - E_1^+}{2}t - \sin^2\frac{E_2^+ - E_1^-}{2}t\right\}. \quad (4.32)$$

Note that two last terms here are suppressed by the presence of $\cos\phi \ll 1$.

If we also account for a rather general condition $2\mu B_\perp \ll p$ then the following approximation can be used for the neutrino energies

$$E_i^\pm \approx \sqrt{m_i^2 + p^2 \pm 2\mu_i p B_\perp\left(1 + \frac{m_i^2}{2p^2}\right)} \approx p\sqrt{1 + \frac{m_i^2}{p^2} \pm \frac{2\mu_i B_\perp}{p}} \approx p + \frac{m_i^2}{2p} \pm \mu_i B_\perp. \quad (4.33)$$

Finally, for the neutrino oscillations probability in the flavour basis we get

$$P_{\nu_{eL}\to\nu_\mu}(t) \approx \sin^2 2\theta\sin^2\frac{\Delta m^2}{4p}t + \frac{1}{2}\left(\sin^2\frac{\mu_2 - \mu_1}{2}B_\perp t + \sin^2\frac{\mu_2 + \mu_1}{2}B_\perp t\right)\sin^2 2\theta\cos\frac{\Delta m^2}{2p}t$$
$$- \frac{1}{4}\sin^2 2\theta\left(\sin^2\mu_1 B_\perp t + \sin^2\mu_2 B_\perp t\right), \quad (4.34)$$

where $\Delta m^2 \equiv m_2^2 - m_1^2$.

It should be emphasized that, as it follows from the above derivations, the obtained probability $P_{\nu_{eL}\to\nu_\mu}(t)$ accounts for the transitions from the initial left-handed electron neutrino to the final muon neutrino that can be in both left- and right-handed states. Note that in the mass basis the transition $\nu_{1L} \to \nu_{2R}$ is not possible when $\mu_{12} = 0$ (see eq.(2.26)).

Consider the difference $\Delta P(t) = P_{\nu_{eL}\to\nu_\mu} - P_{\nu_{eL}\to\nu_{\mu L}}$ of the probability $P_{\nu_{eL}\to\nu_\mu}(t)$ given by (4.36) and the usual result

$$P_{\nu_{eL}\to\nu_{\mu L}}(t) = \sin^2 2\theta\sin^2\frac{\Delta m^2}{4E_\nu}t \quad (4.35)$$

for the neutrino flavour transition $\nu_{eL} \to \nu_{\mu L}$. Obviously, this difference is just a probability for the transition $P_{\nu_{eL}\to\nu_{\mu R}}$, and we get

$$P_{\nu_{eL}\to\nu_{\mu R}} = \frac{1}{2}\left(\sin^2\frac{\Delta\mu_-}{2}B_\perp t + \sin^2\frac{\Delta\mu_+}{2}B_\perp t\right)\sin^2 2\theta\cos\frac{\Delta m^2}{2p}t$$
$$- \frac{1}{4}\sin^2 2\theta\left(\sin^2\mu_1 B_\perp t + \sin^2\mu_2 B_\perp t\right), \quad (4.36)$$

where the notations are used $\Delta\mu_\pm = \mu_2 \pm \mu_1$.

Here we recall that the result (4.36) is obtained within an approach that implies the description of the neutrino spin states by using the spin operator (4.6) that commutes with the neutrino dynamical Hamiltonian. Contrary to what is usually obtained (see, for instance, in [2]) that the transitions $\nu_{eL} \to \nu_{\mu R}$ in $B_\perp$ are not possible in the case of zero transition magnetic moments, from (4.36) we show that the discussed transitions can proceed even in the case when the transition magnetic moments are zero. This predicted new phenomenon can have important phenomenological consequences in implications of neutrino transitions in magnetized astrophysical media.

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